Field Trips and Fund-Raisers

Introducing Fractions

Catherine Twomey Fosnot

*first*hand
An imprint of Heinemann
A division of Reed Elsevier, Inc.
361 Hanover Street
Portsmouth, NH 03801–3912
firsthand.heinemann.com

Harcourt School Publishers
6277 Sea Harbor Drive
Orlando, FL 32887–6777
www.harcourtschool.com

ISBN 0-325-01023-4

ISBN 0-153-60575-8

Offices and agents throughout the world

Library of Congress Cataloging-in-Publication Data
CIP data is on file with the Library of Congress

Printed in the United States of America on acid-free paper

11 10 09 08 07 ML 1 2 3 4 5 6

Acknowledgements

Photography

Herbert Seignoret
Mathematics in the City, City College of New York

Contents

Unit Overview

The focus of this unit is the development of fractions. It begins with the story of a class field trip. The class split into four groups and each group was given submarine sandwiches to share for lunch. Upon returning from their trip, the students quarreled over whether some received more to eat than others.

Note: This unit begins with the fair sharing of submarine sandwiches on a field trip. This context was field-tested by the Freudenthal Institute and the University of Wisconsin, under the direction of Thomas Romberg and Jan de Lange, in preparation for the writing of *Mathematics in Context: Some of the Parts* (van Galen, Wijers, Burrill, and Spence 1997) and it has been researched and written about extensively as it is used in this unit by Fosnot and Dolk (2002).

This story context sets the stage for a series of investigations in this unit. First, students investigate whether the situation in the story was fair—was the quarreling justified?—thereby exploring the connection between division

The Landscape of Learning

BIG IDEAS

- Fractions are relations—the size or amount of the whole matters
- Fractions may represent division with a quotient less than one
- With unit fractions, the greater the denominator, the smaller the piece is
- Pieces don't have to be congruent to be equivalent
- For equivalence, the ratio must be kept constant

STRATEGIES

- Using landmark unit fractions or using common fractions
- Using decimal and/or percentage equivalents
- Using a ratio table as a tool to make equivalent fractions
- Using multiplication and division to make equivalent fractions
- Using a common whole to compare fractions

MODELS

- Fair sharing
- Ratio table
- Measurement
- Fraction bars
- Double number line

and fractions, as well as ways to compare fractional amounts. As the unit progresses, students explore other cases to determine fair sharing and then make a ratio table to ensure fair sharing during their future field trips. They also design a 60k bike course for a fund-raiser, a context that introduces a bar model for fractions and provides students with another opportunity to explore equivalent fractions.

Several minilessons for division of whole numbers using simplified equivalents are also included in the unit. These are structured using strings of related problems as a way to more explicitly guide learners toward computational fluency with whole-number division and to build a connection to equivalent fractions.

Note: The context for this unit assumes that your students have had prior experience with arrays for multiplication and division, as well as partitive and quotative division with whole numbers. If this is not the case, you might find it helpful to first use the units *The Teachers' Lounge* and *Minilessons Throughout the Year: Multiplication and Division.*

The Mathematical Landscape

Have you ever watched students trying to fold a strip of paper into thirds? Because this is so difficult to do, they often make three equal pieces first and then snip off the sliver of the strip that remains and declare they have made thirds! Of course they have changed the whole, so they do not have 1/3 of the original strip, just three congruent pieces with part of the strip thrown away! Constructing the idea that fractions are relations and thus the size or the amount of the whole matters is an important big idea underlying an understanding of fractions. The conception that removing a small piece doesn't matter results from fractions being taught as a shading activity of part-whole relations divorced from division. Research by Leen Streefland (1991) of the Freudenthal Institute in the Netherlands has shown that learners would do better if they started exploring fractions in more realistic fair-sharing contexts, such as one candy bar shared among three people. No child is willing to discard a piece in this context!

BIG IDEAS

This unit is designed to encourage the development of some of the big ideas underlying fractions:

- ***fractions are relations—the size or amount of the whole matters***
- ***fractions may represent division (both partitive and quotative forms) with a quotient less than one***
- ***with unit fractions, the greater the denominator, the smaller the piece is***
- ***pieces don't have to be congruent to be equivalent***
- ***for equivalence, the ratio must be kept constant***

❖ *Fractions are relations—the size or amount of the whole matters*

Fractions are relations: a ratio of part to whole (3 parts out of 4) or a rate (3 sandwiches for 4 people). Fractions can also be operators. For example, $\frac{3}{5}$ could actually be more than $\frac{4}{5}$, if we are talking about $\frac{3}{5}$ *of* 15 versus $\frac{4}{5}$ *of* 10! Constructing the idea that fractions are relations and that the size or amount of the whole matters is a critical step in understanding fractions.

❖ *Fractions may represent division (both partitive and quotative forms) with a quotient less than one*

Just as there are different ways of thinking about division, there are different ways of thinking about fractions. For example, 12 cookies shared among 3 children is $\frac{12}{3} = 4$. This example is a rate, a partitive form of division: 12 for 3, or 4 for 1. The problem: 12 cookies, 3 to a bag, how many bags? can also be represented as $\frac{12}{3}$. But this example requires us to think about measurement. How many times does a group of 3 fit into 12? This is a quotative form of division. Fractions can be thought of similarly. Three submarine sandwiches shared among 4 people (partitive) is $\frac{3}{4}$, or $\frac{3}{4}$ for 1. In this case, $\frac{3}{4}$ can also be thought of as how many times a bar of 4 units fits into a bar of 3 units (quotative). Not once—only $\frac{3}{4}$ of the 4 fits.

When fractions are developed with fair-sharing division situations, it is easier for learners to construct the big idea that multiplication and division are related to fractions: 3 subs shared among 5 children results in each child getting $\frac{1}{5}$ of each sub. Because there are 3 subs, everyone gets $3 \times \frac{1}{5}$, or $\frac{3}{5}$. The idea of fractions as division is an important idea on the landscape. To deeply understand fractions, learners need to generalize the partitive and quotative relations: 1 candy bar shared among 8 children is equivalent to 1 out of 8 parts. Another way to think about this is to think of how 3 subs shared with 5 people (3 divided by 5) results in $\frac{3}{5}$ of one sub. Learners do not need to know the terms *partitive* and *quotative*, but they do need to know that 3 things shared among 5 people results in $\frac{3}{5}$. The slash symbol between the numerator and denominator is just a symbol to represent division.

❖ *With unit fractions, the greater the denominator, the smaller the piece is*

Students initially may think the reverse—that unit fractions with greater denominators represent greater amounts—because they attempt to generalize their knowledge of whole number to fractions. For example, they reason that since 8 is greater than 7, $\frac{1}{8}$ must be greater than $\frac{1}{7}$. When students are introduced to fractions in fair-sharing contexts, it is easy for them to understand that the greater the denominator, the smaller the piece. When eight people share a pizza, each piece is smaller than when seven people share it.

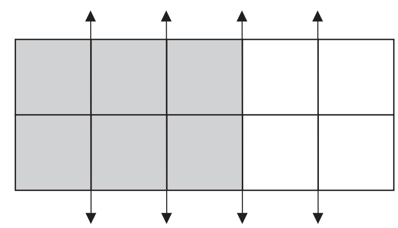
❖ *Pieces don't have to be congruent to be equivalent*

Fair-sharing contexts also provide learners with opportunities to explore how fractional parts can be equivalent without necessarily being congruent. They may look different but still be the same amount. For example, a square can be cut into two triangular halves (diagonally) or two rectangular halves (vertically). The pieces may look different, but the areas are equivalent.

❖ *For equivalence, the ratio must be kept constant*

Equivalence of fractions is often a difficult concept for students to understand. Traditionally, learners have been taught to make equivalent fractions by multiplying or dividing by one (in the form of $\frac{2}{2}$, $\frac{3}{3}$, etc.). Even when learners successfully use this strategy and can parrot back that they are multiplying by one, they may not be convinced that the fractions are really equivalent. Understanding that $\frac{6}{10}$ is not $\frac{3}{5}$ doubled requires that learners understand the implicit 2 for 1 ratio. Imagine a rectangle cut into tenths with six out of the ten shaded, as shown below. Establishing equivalence requires that every two become a new part (2 for 1). Then there are fifths (cut with the arrows) instead of tenths and only three shaded parts instead of six.

$\frac{3}{5} = \frac{6}{10}$



Fair sharing is a very helpful context for exploring equivalence because it is often easier for students to work with than the part-whole model. For example, it is much easier for learners to understand that 3 subs shared among 5 children is an equivalent situation to 6 subs shared among 10 children. If you double the number of people, you better double the number of subs! This is an example of keeping the ratio constant.

STRATEGIES

As you work with the activities in this unit, you will also notice that students will use many strategies to derive answers and to compare fractions. Here are some strategies to notice:

- ***using landmark unit fractions or using common fractions***
- ***using decimal and/or percentage equivalents***
- ***using a ratio table as a tool to make equivalent fractions***
- ***using multiplication and division to make equivalent fractions***
- ***using a common whole to compare fractions***

❖ *Using landmark unit fractions or using common fractions*

When students are faced with fair-sharing situations, they will usually mathematize them in one of two ways. For example, when sharing subs they may use unit fractions (fractions with numerators of one) and cut one sub with landmark amounts first (such as $\frac{1}{2}$ or $\frac{1}{4}$) and continue with each sub and then try to add the amounts up (in the history of fractions this strategy is very similar to how early Egyptians thought about fractions). To represent sharing 3 subs among 5 people, students will usually do the following: Cut each sub in half. Everyone gets $\frac{1}{2}$. Cut the remaining $\frac{1}{2}$ up into fifths. This produces $\frac{1}{10}$ of a sub for each person. So everyone gets $\frac{1}{2} + \frac{1}{10}$.

The second strategy is different. It produces common fractions—in which the numerator is not one. Here each sub is cut into fifths at the start, since there are five people. Now everyone gets $3 \times \frac{1}{5}$, or $\frac{3}{5}$ of a sub.

Each of these strategies brings learners to different roadblocks. In both cases learners may struggle to determine what the total amount is. When using unit fractions, they must figure out how to add fractions with different denominators, and they must determine what to call $\frac{1}{5}$ of $\frac{1}{2}$. They are faced with the idea that the whole matters. They need to know what the whole is in order to name the part: is the fifth the whole, or is the sub the whole? is the sliver a fifth or a tenth? When using common fractions, students end up with $3 \times \frac{1}{5}$ and are faced with the relationship of division and multiplication: 3 subs divided by 5 people produces 1 divided by 5, times 3. They also encounter the relationship between partitive (fair sharing) and quotative (part-whole) representations of fractions and can be pushed to generalize about these relations.

❖ *Using decimal and/or percentage equivalents*

Some students may recognize the "3 subs for 5 people" situation as division and use the long division algorithm or grab a calculator. This move will result in a decimal quotient of 0.6. A few students may even turn this into a percentage equivalent and report that everyone gets 60 percent. In these cases, it is important to have them develop a justification for the equivalence using pictures of the sub. Can they show that 0.6, or 60 percent, is equivalent to 3/5? Justifying equivalence in a drawing (as on page 7) may become a roadblock. As you progress through this unit, look for opportunities for rich discussion around these strategies and big ideas and help students work through the roadblocks.

❖ *Using a ratio table as a tool to make equivalent fractions*

The fair-sharing situations in this unit help to generate use of the ratio table as a tool to make equivalent fractions. Students will notice patterns and develop strategies to make equivalent fractions. The first strategy you will most likely see is a doubling strategy. If you double the amount to be shared and you double the number of people, the result is the same. This strategy is very helpful in some cases, such as $\frac{3}{5} = \frac{6}{10} = \frac{?}{20}$, but it is not sufficient for all cases, for example when numbers are not as friendly. Students can also use the ratio table to keep rates equivalent by adding numerators and adding denominators. Three subs for 4 people and 6 subs for 8 people can be used to derive 9 subs for 12 people.

❖ *Using multiplication and division to make equivalent fractions*

Eventually students will construct the strategy of using multiplication and division to make equivalent fractions. For example, to find an equivalent for 3/5 with a denominator of 10, students must multiply the 3 by 2, since 10 divided by 5 equals 2, to produce 6; or students can divide the 10 by the 5 to produce 2 and then multiply the 3 by 2 to produce 6. Because fractions can be thought of as division and simplifying fractions is an important strategy for making equivalents, it is often a very efficient strategy to use even when dividing whole numbers. For example, 300 divided by 12 can be simplified to 100 divided by 4. The simplified version can be done mentally! This unit was carefully crafted and field-tested to ensure that simplifying is specifically addressed, both for fractions and for division with whole numbers. Not only will a discussion on this idea come up as students explore fair-sharing situations, but minilessons are also included to support the use of this strategy for whole-number division.

❖ *Using a common whole to compare fractions*

Comparing fractions creates another hurdle for students. How can they make a common whole to compare them? Initially, many numbers that are not common multiples may be tried as denominators. It is important to let students grapple with this problem. Eventually they will gravitate toward the realization that a common whole can be made by finding a common multiple. While a common multiple may be the result of their process, it is not important to name it as a common multiple at this time. By allowing their struggle and supporting their quest to find a common whole, you give students the opportunity to "own" the solution when they finally hit on the idea of common denominators.

MATHEMATICAL MODELING

Several mathematical models are developed in this unit, but only two are being introduced for the first time: the ratio table and the measurement model. With the ratio table model, students are supported to envision equivalent fair-sharing relationships. As the unit progresses, the ratio table model is used to support the development of various strategies for finding equivalent rates. Then the measurement model is introduced and equivalent fractions are explored with fraction bars and number lines.

Models go through three stages of development (Gravemeijer 1999; Fosnot and Dolk 2002):

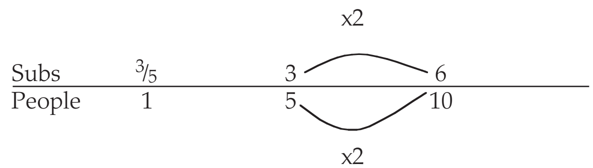
- ❖ ***model of the situation***
- ❖ ***model of students' strategies***
- ❖ ***model as a tool for thinking***

❖ *Model of the situation*

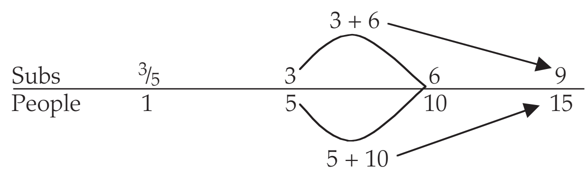
Initially models grow out of modeling the situation—in this unit, the ratio table emerges as a chart to ensure fair sharing during future field trips. The measurement model emerges as students design a 60k bike course.

❖ *Model of students' strategies*

Students benefit from seeing the teacher model their strategies. Once a model has been introduced as a model of the situation, you can use it to model students' strategies as they explain their thinking. If a student says, "I doubled the numerator and doubled the denominator," draw the following:



If a student says, "I added the 5 and 10 to make 15, so I added the 3 and 6 to make 9," draw the following:



Representations like these give students a chance to discuss and envision each other's strategies.

❖ *Model as a tool for thinking*

Eventually learners become able to use the model as a tool to think with—they will be able to use it as a tool to prove and explore their ideas about proportional reasoning. Ratio tables can be presented as t-charts:

Number of Subs	Number of People
$\frac{3}{5}$	1
3	5
6	10
9	15

Measurement models become number lines where fractions can be placed as numbers. In time, this model is very helpful for addition and subtraction of fractions. Although operations are not the focus of this unit, you may find some students beginning to explore that topic.

Many opportunities to discuss these landmarks will arise as you work through this unit. Look for moments of puzzlement. Don't hesitate to let students discuss their ideas and check and recheck their strategies. Celebrate their accomplishments—they are young mathematicians at work!A graphic of the full landscape of learning for fractions, decimals, and percents, with a box highlighting the focus of this unit, is provided on page 11. The purpose of the graphic is to allow you to see the longer journey of your students' mathematical development and to place your work with this unit within the scope of this long-term development. You may also find it helpful to use this graphic as a way to record the progress of individual students for yourself. Each landmark can be shaded in as you find evidence in a student's work and in what the student says—evidence that a landmark strategy, big idea, or way of modeling has been constructed. In a sense, you will be recording the individual pathways your students take as they develop as young mathematicians!

References and Resources

Dolk, Maarten and Catherine Twomey Fosnot. 2005. *Multiplication and Division Minilessons, Grades 3–5.* CD-ROM with accompanying facilitator's guide by Antonia Cameron, Carol M. Mosesson Teig, Sherrin B. Hersch, and Catherine Twomey Fosnot. Portsmouth, NH: Heinemann.

———. 2006a. *Fostering Children's Mathematical Development, Grades 5–8: The Landscape of Learning.* CD-ROM with accompanying facilitator's guide by Sherrin B. Hersch, Catherine Twomey Fosnot, and Antonia Cameron. Portsmouth, NH: Heinemann.

———. 2006b. *Sharing Submarine Sandwiches: A Context for Fractions, Grades 5–8.* CD-ROM with accompanying facilitator's guide by Sherrin B. Hersch, Catherine Twomey Fosnot, and Antonia Cameron. Portsmouth, NH: Heinemann.

Fosnot, Catherine Twomey and Maarten Dolk. 2002. *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents.* Portsmouth, NH: Heinemann.

Gravemeijer, Koeno P.E. 1999. How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–77.

Hersh, Reuben. 1997. *What is Mathematics, Really?* London: Oxford University Press.

Streefland, Leen. 1991. *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research.* Dordrecht, Netherlands: Kluwer.

van Galen, Frans, Monica Wijers, Gail Burrill, and Mary S. Spence. 1997. *Mathematics in Context: Some of the Parts.* Orlando, FL: Holt, Rinehart, and Winston.